## Easy Logarithms for COP400

Logarithms have long been a convenient tool for the simplification of multiplication, division, and root extraction. Many assembly language programmers avoid the use of logarithms because of supposed complexity in their application to binary computers. Logarithms conjure up visions of time consuming iterations during the solution of a long series. The problem is far simpler than imagined and its solution yields, for the applications programmer, the classical benefits of logarithms:

1) Multiplication can be performed by a single addition.
2) Division can be performed by a single subtraction.
3) Raising a number to a power involves a single multiply.
4) Extracting a root involves a single divide.

When applied to binary computer operation logarithms yield two further important advantages. First, a broad range of values can be handled without resorting to floating point techniques (other than implied by the characteristic). Second, it is possible to establish the significance of an answer during the body of a calculation, again, without resorting to floating point techniques.
Implementation of base ${ }_{10}$ logarithms in a binary system is cumbersome and unnecessary since logarithmic functions can be implemented in a number system of any base. The techniques presented here deal only with logarithms to the base $_{2}$.
A logarithm consists of two parts: an integer characteristic and a fractional mantissa.


FIGURE 1. The Logarithmic Function and Some Example Values

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In Figure 1 some points on the logarithmic curve are identified and evaluated to the base ${ }_{2}$. Notice that the characteristic in each case represents the highest even power of 2 contained in the value of $X$. This is readily seen when binary notation is used.

| $\mathrm{X}_{10}$ |  | 23 | $\begin{aligned} & X_{2} \\ & \mathbf{2}^{2} \end{aligned}$ |  |  | $\log _{2} X$ <br> Characteristic | $\log _{2} X \text { Where } X=$ $\text { Even Power of } 2$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 3 | 0 | 0 | 0 | 4 | 1 | 1 |  |
| 4 | 0 | 0 | 4 | 0 | 0 | 2 | 010.0000 |
| 8 | 0 | 4 | 0 | 0 | 0 | 3 | 011.0000 |
| 10 | 0 | 4 | 0 |  | 0 | 3 |  |

## FIGURE 2. Identification of the Characteristic

In Figure 2 each point evaluated in Figure 1 has been repeated using binary notation. An arrow subscript indicates the highest even power of 2 appearing in each value of $X$. Notice that in $X=3$ the highest even power of 2 is $2^{1}$. Thus the characteristic of the $\log _{2} 3$ is 1 . Where $X=10$ the characteristic of the $\log _{2} 10$ is 3 .
To find the $\log _{2} \mathrm{X}$ is very easy where X is an even power of 2. We simply shift the value of $X$ left until a carry bit emerges from the high order position of the register. This procedure is illustrated in Figure 3. This characteristic is found by counting the number of shifts required and subtracting the result from the number of bits in the register. In practice it is easier to being with the number of bits and count down once prior to each shift.

| Counter for <br> Characteristic | Value of X in Binary |  |  |
| :---: | :---: | :---: | :--- |
| 1000 | 0000 | 1000 | Initial |
| 0111 | 0001 | 0000 | First Shift |
| 0110 | 0010 | 0000 | Second Shift |
| 0101 | 0100 | 0000 | Third Shift |
| 0100 | 1000 | 0000 | Fourth Shift |
| 0011 | 0000 | 0000 | Fifth Shift |
| Characteristic | Mantissa | Final |  |
| 011.0000 | 0000 | $\log _{2} \mathrm{X}=3.00$ |  |

FIGURE 3. Conversion to Base ${ }_{2}$ Logarithm by Base Shift

Examination of the final value obtained in Figure 3 reveals no bits in the mantissa. The value 3 in the characteristic, however, indicates that a bit did exist in the $2^{3}$ position of the original number and would have to be restored in order to reconstruct the original value (antilog).

The log of any even power of 2 can be found in this way:

| Decimal | Binary | $\log _{2}$ |
| :---: | :---: | :---: |
| 128 | 10000000 | 0111.00000000 |
| 64 | 01000000 | 0110.00000000 |
| 32 | 00100000 | 0101.00000000 |
| 4 | 00000100 | 0010.00000000 |
| 2 | 00000010 | 0001.00000000 |
| 1 | 00000001 | 0000.00000000 |

FIGURE 4. Base $_{2}$ Logarithms of Even Powers of 2

A simple flow chart, and program, can be devised for generating the values found in the table and, as will be apparent, a straight line approximation for values that are not even powers of 2. The method, as already illustrated in Figure 3, involves only shifting a binary number left until the most significant bit moves into the carry position. The characteristic is formed by counting. Since a carry on each successive shift will yield a decreasing power of 2 , we must start the characteristic count with the number of bits in the binary value (x) and count down one each shift.



The program shown develops the $\log _{2}$ of any even power of 2 by shifting and testing as previously described. Examine what happens to a value of $X$ that is not an even power of 2 . In Figure 7, the number 25 is converted to a base 2 log .

$$
\begin{gathered}
25_{10}=00011002_{2} \\
\quad \text { Shift left until carry }=1
\end{gathered}
$$

Characteristic Carry Mantissa $\mathbf{L o g}_{2}$
$0100 \quad 1 \quad 100100000100.10010000$

## Figure 7. Straight Line Approximation of Base $\mathbf{L}_{2}$ Log

The resulting number when viewed as an integer characteristic and a fractional mantissa is $4.5625_{10}$. The fraction 0.5625 is a straight line approximation of the logarithmic curve between the correct values for the base 2 logs of $2^{4}$ and $2^{5}$. The accuracy of this approximation is sufficient for many applications. The error can be corrected, as will be seen later in this discussion, but for now let's look at the problem of exponents or the conversion to an antilog.

To reconstruct the original value of $X$, find the antilog, requires only restoration of the most significant bit and then its alignment with the power of 2 position indicated by the characteristic. In the example, approximation $\left(\log _{2} 25=\right.$ 0100.1001) restoration of MSB can be accomplished by shifting the mantissa (only) one position to the right. In the process a one is shifted into the MSB position.

| Approximation of $\log _{2} \mathbf{X}$ | Restoration of MSB |
| :---: | :---: |
| Char. Mantissa | Char. Mantissa |
| 0100.10010000 | 0100.11001000 |

The value of the characteristic is 4 so the mantissa must be shifted to the right until MSB is aligned with the $2^{4}$ position.

$$
\begin{array}{llllllll}
2^{7} & 2^{6} & 2^{5} & 2^{4} & 2^{3} & 2^{2} & 2^{1} & 2^{0}
\end{array}
$$

The completion of this operation restores the value of $X$ $(X=25)$ and is the procedure used to find an antilog. Figure 8 is a flow chart for finding an antilog using this procedure. Ths implementation in source code is shown in Figure 9.


FIGURE 8. Flow Chart for Conversion to Antilog

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LOGS
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FIGURE 9 TL/DD/6942-6
FIGURE 9
Using the linear approximation technique just described, some error will result when converting any value of $X$ that is not an even power of 2 .
Figure 10 contains a table of correct base 2 logarithms for values of $X$ from 1 through 32 along with the error incurred for each when using linear approximation. Notice that no error results for values of $X$ that are even powers of 2 . Also notice that the error incurred for multiples of even powers of 2 of any given value of X is always the same.

| Value of $\mathbf{X}$ | Error |
| :---: | :---: |
| 5 | 0.12 |
| $2 \times 5=10$ | 0.12 |
| $4 \times 5=20$ | 0.12 |
| 3 | 0.15 |
| $2 \times 3=6$ | 0.15 |
| $4 \times 3=12$ | 0.15 |
| $8 \times 3=24$ | 0.15 |

$\left.\begin{array}{ccccc}\mathbf{X} & & & & \\ \hline \text { Hexadecimal } \\ \text { Log Base }\end{array} \quad \begin{array}{c}\text { Linear } \\ \text { Approximation } \\ \text { of Log Base 2 }\end{array}\right)$

FIGURE 10. Error Incurred by Linear Approximation of Base 2 Logs

An error that repeats in this way is easily corrected using a look-up table. The greatest absolute error will occur for the least value of $X$ not an even power of $2, X=3$, is about $8 \%$. A 4 point correction table will eliminate this error but will move the greatest uncompensated error to $X=9$ where it
will be about $4 \%$. This process continues until at 16 correction points the maximum error for the absolute value of the logarithm is less than 1 percent. This can be reduced to 0.3 percent by distributing the error. Interpolated error values are listed in Figure 10 and are repeated in Figure 11 as a binary table.

| High Order <br> 4 Mantissa <br> Bits | Binary <br> Correction <br> Value | Hexadecimal <br> Correction <br> Value |
| :---: | :---: | :---: |
| 0000 | 00000000 | 00 |
| 0001 | 00001001 | 09 |
| 0010 | 00001101 | 03 |
| 0011 | 00010001 | 11 |
| 0100 | 00010101 | 15 |
| 0101 | 00010110 | 16 |
| 0110 | 00010110 | 16 |
| 0111 | 00010110 | 16 |
| 1000 | 00010101 | 15 |
| 1001 | 00010100 | 14 |
| 1010 | 00010010 | 12 |
| 1011 | 00010000 | 10 |
| 1100 | 00001101 | 00 |
| 1101 | 00001010 | 0 A |
| 1110 | 00000110 | 06 |
| 1111 | 00000010 | 02 |
| FIGURE 11. Correction Table for |  |  |
| L $_{\mathbf{2}}$ X Linear Approximations |  |  |

Notice in Figure 10 that left justification of the mantissa causes its high order four bits to form a binary sequence that always corresponds to the proper correction value. This works to advantage when combined with the COP400 LQID instruction. LQID implements a table look-up function using the contents of a memory location as the address pointer. Thus we can perform the required table look-up without disturbing the mantissa.

Figure 12 is the flow chart for correction of a logarithm found by linear approximation. Figure 13 is its implementation in COP400 assembly language. Notice that there are two entry points into the program. One is for correction of logs (LADJ:), the other is for correction of a value prior to its conversion to an antilog (AADJ:).


FIGURE 12. Flow Chart for Correction of a Value Found by Straight Line Approximation

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COP CROSS ASSEMBLER PAGE: }
LOGS
\begin{tabular}{|c|c|c|}
\hline 110 & . FORM & \(\rightarrow\) ADJUST VALUE OF LOGARITHM \\
\hline 111 & & \\
\hline 112 & & LOCAL \\
\hline
\end{tabular}
THE FOLLOWING TABLE IS USED DURING THE CORRECTION OF VALUES
FOUND BY STRAIGHT LINE APPROXIMATION. IT IS PLACED HERE IN
ORDER TO ALIGN ITS BEGINNING ELEMENT WITH A ZERO ADDRESS AS REQUIRED BY THE LQID INSTRUCTION.
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THE FOLLOWING SUBROUTINE ADJUSTS THE VALUE OF A BASE 2
LOGARITHM FOUND BY STRAIGHT LINE APPROXIMATION. THE CORRECTION TERMS ARE TAKEN FROM THE TABLE ABOVE. THE SUBROUTINE HAS 2 ENTRY POINTS
ADJ: - ADJUSTS A VALUE DURING CONVERSION TO A LOG
AADJ: - ADJUSTS A VALUE DURING CONVERSION TO ANTILOG
THE CARRY FLAG IS SET UPON ENTRY TO DISTINGUISH BETWEEN LOG
( \(C=1\) ) AND ANTILOG ( \(C=0\) ) CONVERSIONS. DURING A LOGARITHM
CONVERSION THE VALUE FOUND IN THE ABOVE TABLE IS ADDED TO
THE MANTISSA. DURING AN ANTILOG CONVERSION THE VALUE FOUND IN THE ABOVE TABLE IS SUBTRACTED FROM THE MANTISSA.
\begin{tabular}{ll} 
& \\
030 & 32 \\
031 & F 3 \\
032 & 22 \\
033 & 05 \\
034 & 07 \\
035 & 05 \\
036 & 37 \\
037 & 06 \\
038 & 00 \\
039 & 52
\end{tabular}
AAD
ADJ:
\(\begin{array}{ll}\text { RC } \\ J P & \$ L\end{array}\)
\$LD
\$LD
03
TBL
C=O FOR ANTILOG
CONVERSION.
C = FOR LOG 2 AD
MOVE ADDRESS POINTER BACK ONE LOCATION.
LOAD CONTENTS OF HI MANTISSA AND STORE IT IN THE LO ORDER OF THE TEMP MEMORY LOCATION SET TABLE POINTER (ACC) TO TABLE ADDRESS
COP CROSS ASSEMBLER PAGE: 5 LOGS
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline 152 & 03A & BF & \multirow{5}{*}{\$GTM:} & LQID & & \multirow[t]{3}{*}{LOAD CORRECTION VALUE TO Q. TRANSFER Q REGISTER CONTENTS TO MEMORY.} \\
\hline 153 & 03 B & 332 C & & \multicolumn{2}{|l|}{CQMA} & \\
\hline 154 & 030 & 04 & & XIS & & \\
\hline 155 & 03F & 07 & & \multicolumn{3}{|l|}{XDS} \\
\hline 156 & 03F & 20 & & \multicolumn{2}{|l|}{SKC} & ; ANTILOG? \\
\hline 157 & 040 & 80 & & JSRP & COMP & ; YES - COMPLIMENT. \\
\hline 158
159 & 041 & 98 & \$ADD: & JSRP & ADRO & \begin{tabular}{l}
; ADD CORRECTION VALUE \\
; TO MANTISSA.
\end{tabular} \\
\hline 160 & 042 & 35 & & LD & 03 & ; SET POINTER TO \\
\hline 161 & 043 & 48 & \$LST: & RET & ; CHAR & \\
\hline 162 & & & & & & RETURN. \\
\hline \multicolumn{7}{|l|}{163} \\
\hline 164 & & & \multicolumn{4}{|l|}{; 2 ROUTINES ARE CALLED FROM THE SUBROUTINE PAGE BY THIS} \\
\hline 165 & & & \multicolumn{4}{|l|}{; PROGRAM: COMP, ADRO} \\
\hline \multicolumn{7}{|l|}{166} \\
\hline 167 & & 0020 & & \multicolumn{3}{|l|}{V1 = TPLS\&OFF} \\
\hline 168 & & 0002 & & \multicolumn{3}{|l|}{TBL \(=\mathrm{V}_{1} / 16\)} \\
\hline
\end{tabular}
```


## Subroutines Used by the Log and Antilog Programs





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